School of Mathematics and Statistics
Faculty of Science, Technology, Engineering and Mathematics

The Open University

2026 PhD Projects

Project title	Substitutions on countable alphabets
Principal supervisor	Dan Rust
Second supervisor	Vasiliki Evdoridou
Discipline	Pure
Research area/keywords	Topology, Dynamics, Aperiodic Order
Suitable for	Full time applicants, Part time applicants

Project background and description

In this project, we will investigate combinatorial, topological and dynamical properties of a new class of metric space; shift spaces of substitutions on infinite alphabets.

The simplest non-trivial substitution is the Fibonacci substitution defined by the rule $0 \mapsto 01,\ 1 \mapsto 0$. When iterated, one generates longer and longer words (and a limit word):

$$0 \mapsto 01 \mapsto 010 \mapsto 01001 \mapsto \cdots \mapsto 0100101001001010 \cdots .$$

A generalisation of the Fibonacci substitution on a three letter alphabet is the *Tribonacci* substitution $0 \mapsto 01,\ 1 \mapsto 02,\ 2 \mapsto 0$, and a generalisation on an infinite alphabet is the *Infibonacci* substitution $i \mapsto 0\,(i+1),\ \infty \mapsto 0\,\infty$, which was first studied 20 years ago [1]. A typical question that one asks is what is the frequency of each of the letters in the limit word. For example, using some linear algebra, one can show that the 0s in the Fibonacci example have frequency $1/\phi$, where $\phi \approx 1.618$ is the golden ratio, and in the Infibonacci example it has frequency $1/2$. Other questions include whether the limit word is periodic, if every subword that appears does so infinitely often, or whether any cubes appear in the word (subwords u such that uuu also appears).

Even for as basic a question as determining letter-frequencies, no method is currently known for substitutions on a general infinite alphabet, and we were only recently able to identify conditions that imply the existence of letter-frequencies [2]. The key to that work was imposing a topological condition on the substitution; namely a continuous substitution on a compact alphabet. Such substitutions generate dynamical systems called *self-induced systems* [3]. In this setting, techniques from topology, analysis and operator theory are extremely powerful.

In this project, we will restrict the alphabet even further to countable (compact) alphabets, and develop a framework for studying the resulting limit words and dynamical systems. It is expected that this setting allows for methods from even more varied areas of mathematics to be put to use, including graph theory, combinatorial set theory, and combinatorics on words.

Background reading/references

[1] S. Ferenczi, *Substitution dynamical systems on infinite alphabets*, Ann. Inst. Fourier (Grenoble), 56:2315–2343 (2006).

[2] N. Mañibo, D. Rust, J. Walton, *Substitutions on compact alphabets*, J. Lon. Math. Soc., 111(3), e70123 (2024).

[3] F. Durand, N. Ormes, and S. Petite, *Self-induced systems*, J. Anal. Math., 135:725–756 (2018).